Computer Science II

Mid-term Exam

4 March 2009

1. Explain how numbers are stored in the floating point system characterized by (β, t, L, U). What are the two ways of representing any real number in the floating-point system ? Give the definition of unit round.

2. To convert a positive decimal fraction $x < 1$ to its binary equivalent

$$x = (.a_1 a_2 a_3 \ldots)_2$$

begin by writing

$$x = a_1 \cdot 2^{-1} + a_2 \cdot 2^{-2} + a_3 \cdot 2^{-3} + \ldots$$

Based on this use the following algorithm

(a) $x_1 := x$; $j := 1$

(b) While $x_j \neq 0$, do the following
$a_j :=$ Integer part of $2x_j$
$x_{j+1} :=$ Fractional part of $2x_j$
$j := j + 1$
End while

Apply this algorithm to convert $x = 2/3$ to its binary form. Represent $x = 2/3$ in the floating-point system $(\beta, t, L, U) = (2, 6, -6, +5)$ using both chopping and rounding.

3. Consider the second order divided difference $f[x_0, x_1, x_2]$. (a) Prove that the order of the arguments x_0, x_1, x_2 does not affect the value of the divided difference. (b) Prove that

$$f[x_0, x_1, x_2] = \frac{1}{2} f''(\xi)$$

for some ξ between the minimum and maximum of x_0, x_1 and x_2.

4. A variant of Newton's method for root finding called Steffenson's Method uses an approximation to the derivative and is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{D(x_n)}, \quad D(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$$

Let α be a root of $f(x)$ and assume $f'(\alpha) \neq 0$. Write the iteration as $x_{n+1} = \phi(x_n)$ and show that $\alpha = \phi(\alpha)$. Show that this is a second order method, i.e. it converges quadratically.

5. Consider linear interpolation applied to tabular data $[x_i, f(x_i)]$. The function values are not exact since they are rounded to a few decimal places. You have to interpolate for some $x_0 < x < x_1$ given the rounded function values f_0, f_1 which have errors $\epsilon_0 = f(x_0) - f_0$ and $\epsilon_1 = f(x_1) - f_1$. Show that the error of linear interpolation is

$$|e(x)| \leq \frac{h^2}{8} \max_{x_0 \leq t \leq x_1} |f''(t)| + \max\{|\epsilon_0|, |\epsilon_1|\}$$

which is a sum of interpolation error and round-off error.

6. Suppose you have a table of logarithms to base 10 for $1 \leq x \leq 2$. The spacing between x_i is $h = 0.001$ and $\log(x)$ is correctly rounded to five decimal places so that the error in $\log(x)$ is less than $\epsilon = 0.000005$. For linear interpolation, estimate the interpolation error, round-off error and total error. Which error is dominant ? If you have to improve the table to reduce the total interpolation error, how would you do it ? What would be a good strategy to choose h and the number of decimal places so that the total interpolation error is less than a specified upper bound ? Apply the result of previous problem and use $\log(e) \approx 0.434$.

7. Consider the problem of finding a quadratic polynomial $p(x)$ for which

$$p(x_0) = y_0, \quad p'(x_1) = y_1', \quad p(x_2) = y_2$$

with $x_0 \neq x_2$ and $\{y_0, y_1', y_2\}$ the given data. What conditions must be satisfied for such a $p(x)$ to exist and be unique ?

8. Explain what is a cubic interpolating spline. Consider a cubic interpolating spline with the additional boundary conditions

$$s''(x_0) = M_0 = 0, \quad s''(x_n) = M_n = 0$$

Show that

$$\int_{x_0}^{x_n} [s''(x)]^2 dx \leq \int_{x_0}^{x_n} [g''(x)]^2 dx$$

where $g(x)$ is any twice continuously differentiable function that satisfies the interpolating conditions $g(x_i) = y_i$, $i = 0, 1, \ldots, n$.

9. Let $f(x)$ be three times continuously differentiable on $[-\alpha, +\alpha]$ for some $\alpha > 0$, and consider approximating it by the rational function

$$R(x) = \frac{a + bx}{1 + cx}$$

Choose the constants a, b, c so that

$$R^{(j)}(0) = f^{(j)}(0), \quad j = 0, 1, 2$$

Is it always possible to find such an approximation $R(x)$?

10. The iteration

$$x_{n+1} = 2 - (1 + c)x_n + cx_n^3$$

will converge to $\alpha = 1$ for some values of c (provided x_0 is chosen sufficiently close to α). Find the values of c for which this is true ? For what value of c will the convergence be quadratic ?

11. Given the function values $\{x_i, f_i\}$, $i = 0, 1, \ldots, n$, write down the Lagrange form of the interpolating polynomial and explain how it interpolates the given data. For $n = 2$ and with uniformly spaced data, differentiate the Lagrange polynomial to find a formula which approximates the derivative $f'(x_0)$.

12. Explain what is meant by a minimax polynomial approximation. Let $f \in C^2[a, b]$ with $f''(x) > 0$ for $a \leq x \leq b$. If $q_1^*(x) = a_0 + a_1 x$ is the linear minimax approximation to $f(x)$ on $[a, b]$ then show that

$$a_1 = \frac{f(b) - f(a)}{b - a}, \quad a_0 = \frac{1}{2}[f(a) + f(c)] - \frac{1}{2}(a + c)\left[\frac{f(b) - f(a)}{b - a}\right]$$

where c is the unique solution of

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$